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Boundary Integral Formulation for Composite Laminates in Torsion

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The three-dimensional elastic stress state in a general composite laminate under twisting load is given. The analysis is carried out through an integral equation formulation that is numerically solved by the boundary element method. The integral representation of the elastic behavior is deduced by means of the reciprocity theorem applied to the actual response of each ply and the problem's analytical singular fundamental solutions. The interface continuity conditions due to perfect bonding are considered to complete the laminate mathematical model. The method permits the analysis for generally stacked laminates having general shape of the cross section. By virtue of the formulation characteristics, the stress distributions calculated are not affected by a priori assumptions about their nature. Numerical applications are presented, and the results obtained show that the proposed method allows an accurate prediction of the complete elastic response coupled with meaningful computational advantages.

Nomenclature

$\boldsymbol{D}$, $\boldsymbol{X}$	= strain operators
$\boldsymbol{D}_n$	= boundary traction operator
$\boldsymbol{E}$, $\boldsymbol{Q}_1$, $\boldsymbol{Q}_2$	= elasticity matrices
E_{ij}	= elasticity stiffness coefficients
$\boldsymbol{f}_j$	= fundamental solution body forces
l	= laminate length
$\boldsymbol{N}$	= shape function matrix
$\boldsymbol{p}$	= nodal tractions
$\boldsymbol{s}$	= vector of displacements
s_1, s_2, s_3	= displacements in the x_1, x_2, x_3 direction
$\boldsymbol{t}$	= boundary tractions
$\boldsymbol{u}$	= vector of displacement functions
$\boldsymbol{u}_j$, $\boldsymbol{t}_j$	= fundamental solution displacements and tractions
u_1, u_2, u_3	= displacement functions
x_1, x_2, x_3	= coordinate system for the laminate, where x_3 is equal to z
α_1, α_2	= boundary normal direction cosines
Γ_e	= ply section boundary
δ	= nodal displacements
ε	= strain vector
ε_{ij}	= strain components
ε_j, ε_{33j}, σ_j, σ_{33j}	= fundamental solution strains and stresses
θ	= twisting curvature
σ	= stress vector
σ_{ij}	= stress components
Ω	= laminate cross section
Ω_e	= ply cross section

Introduction

THE three-dimensional stress state arising in laminated composites is the main cause responsible for the delamination phenomenon that significantly affects laminate failure. The accurate prediction of this stress state and, in particular, of interlaminar stresses, therefore, is crucial to develop realistic failure criteria for these currently widely employed structural members. Numerous analysts have approached the problem, and different methods have been used to obtain numerical or analytical solutions. A class of laminate solutions was obtained through the finite difference technique,[1–3] which allows the direct solution of the differential equations governing the problem. The popular finite element method was chosen by many authors[4–14] to compute and investigate the laminate elastic response. The finite element solutions available in the literature differ from one another in formulation, in the order of the employed elements, and in the discretization scheme. Many other methods were also used to solve the problem. A boundary-layer theory,[15,16] the perturbation technique,[17] and a solution in the form of a polynomial[18] and Fourier[19] series were proposed, and the relative results are available in the literature. Interesting approaches based on Lekhnitskii's stress potentials[20–23] and Reissner's variational principle[24,25] were proposed and developed. Finally, the force balance method coupled with the minimum complementary energy principle[26,27] was also employed to analyze laminate behavior.

These approaches often satisfy some elasticity relations in an average sense only or they require aprioristic assumptions about the structure of the stress field and, in particular, about interlaminar stresses. Furthermore the results obtained by the aforementioned methods are, in general, relative to the case of uniform axial action or pure bending. Twisting problems have not received extensive attention even if torsion represents a principal mode of behavior in many applications. Some solutions for this load condition are due to Chan and Ochoa,[11,12] who employed a quasi-three-dimensional finite element model, and Yin,[22,23] who determined the torsional interlaminar stresses by a variational method based upon Lekhnitskii's stress functions. The integral equation theory[28,29] seems to be an advantageous approach to the problem. Indeed, the characteristics of integral equation representations allow the pointwise description of the elastic response as required by the laminate problem. Moreover, the solution of boundary integral models proves very attractive from the computational point of view. Integral equation formulations were presented for both three-dimensional[30] and two-dimensional[31,32] anisotropic general problems, but they are not directly applicable to the multilayer laminate problem.

In the present paper, the twisting problem of the laminated beam with arbitrary configuration of the cross section is approached on the basis of an alternative method.[33,34] The problem is formulated by the boundary integral equations. The integral equations governing the behavior of each individual ply are directly deduced by applying the Betti reciprocal work theorem. The model is obtained assuming that the laminate's layers obey a generalized orthotropic law. The exact analytical fundamental solutions of the problem are employed, and a boundary-only model is obtained by transforming domain integrals into boundary integrals. The laminate mathematical model is recovered from the individual ply integral representation by enforcing the continuity conditions of the stresses and displacements along the interfaces. The traction-free boundary conditions

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allow one to obtain the problem-resolving system. By so doing, a formulation that does not require any a priori assumption or approximation is obtained. Solutions obtained by numerically solving the present formulation through the boundary element method (BEM) are presented with the aim of comparing the results with those existing in the literature and thus ascertaining formulation accuracy and efficiency. The complete elastic solution in terms of displacement and stress fields is also given for typical laminates in torsion.

Ply Elasticity Relations

The composite laminate considered consists of generally stacked layers perfectly bonded along the interfaces. The laminate, with length l and cross section Ω of absolutely general shape, is referred to a coordinate system x_1, x_2, $x_3 = z$, as shown in Fig. 1. Each individual layer, having cross section Ω_e of boundary Γ_e, is considered as an homogeneous, linearly elastic, and generalized orthotropic medium. One of the material principal axes of the layer is parallel to the laminate axis x_2. The laminate is subjected to a twisting action applied at the laminate's ends. By virtue of de Saint Venant's principle, sufficiently far from the laminate's ends, the stress field can be assumed to be independent of the z coordinate. For each ply, the displacement field $\boldsymbol{s}$ has components given by[35]

$$s_1 = u_1(x_1, x_2) - \vartheta x_2 z \tag{1a}$$

$$s_2 = u_2(x_1, x_2) + \vartheta x_1 z \tag{1b}$$

$$s_3 = u_3(x_1, x_2) \tag{1c}$$

where the rigid motion terms have been dropped and u_1, u_2, and u_3 are unknown functions of x_1 and x_2 only. To satisfy the interface continuity conditions, the laminate twisting curvature ϑ has to be the same for all layers. For the displacement field given by Eqs. (1), the strain-displacement relations can be written as

$$\begin{bmatrix} e_{11} \\ e_{22} \\ e_{12} \\ e_{31} \\ e_{32} \end{bmatrix} = \begin{bmatrix} \partial/\partial x_1 & 0 & 0 \\ 0 & \partial/\partial x_2 & 0 \\ \partial/\partial x_2 & \partial/\partial x_1 & 0 \\ 0 & 0 & \partial/\partial x_1 \\ 0 & 0 & \partial/\partial x_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -x_2 \\ x_1 \end{bmatrix} \vartheta$$

$$= \boldsymbol{D}\boldsymbol{u} + \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{X} \end{bmatrix} \vartheta = \boldsymbol{\varepsilon} + \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{X} \end{bmatrix} \vartheta \tag{2a}$$

$$e_{33} = 0 \tag{2b}$$

The constitutive equations of each individual ply, referred to the laminate coordinate system, can be conveniently expressed as follows:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \\ \sigma_{31} \\ \sigma_{32} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & 0 & E_{14} & 0 \\ E_{12} & E_{22} & 0 & E_{24} & 0 \\ 0 & 0 & E_{33} & 0 & E_{35} \\ E_{14} & E_{24} & 0 & E_{44} & 0 \\ 0 & 0 & E_{35} & 0 & E_{55} \end{bmatrix} \boldsymbol{\varepsilon}$$

$$+ \begin{bmatrix} E_{14} & 0 \\ E_{24} & 0 \\ 0 & E_{35} \\ E_{44} & 0 \\ 0 & E_{55} \end{bmatrix} \boldsymbol{X}\vartheta = \boldsymbol{E}\boldsymbol{\varepsilon} + \boldsymbol{Q}_1 \boldsymbol{X}\vartheta \tag{3a}$$

$$\sigma_{33} = [E_{16} \quad E_{26} \quad 0 \quad E_{46} \quad 0] \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \\ \varepsilon_{31} \\ \varepsilon_{32} \end{bmatrix} - E_{46} x_2 \vartheta$$

$$= \boldsymbol{Q}_2 \boldsymbol{\varepsilon} - E_{46} x_2 \vartheta \tag{3b}$$

Finally, the equilibrium equations governing the elastic behavior of the individual ply are given in Ω_e by

$$\boldsymbol{D}^T \boldsymbol{\sigma} = \boldsymbol{0} \tag{4a}$$

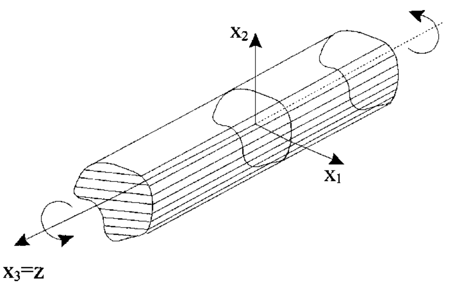

Fig. 1 Laminate configuration and coordinate system.

or in terms of displacements

$$\boldsymbol{D}^T \boldsymbol{E}\boldsymbol{D}\boldsymbol{u} = \boldsymbol{0} \tag{4b}$$

On the boundary Γ_e, the tractions $\boldsymbol{t}$ can be expressed as

$$\boldsymbol{D}_n \boldsymbol{\sigma} = \boldsymbol{t} \tag{5}$$

where, provided that α_1 and α_2 indicate the direction cosines of the outward unit normal to the ply section boundary, one has

$$\boldsymbol{D}_n = \begin{bmatrix} \alpha_1 & 0 & \alpha_2 & 0 & 0 \\ 0 & \alpha_2 & \alpha_1 & 0 & 0 \\ 0 & 0 & 0 & \alpha_1 & \alpha_2 \end{bmatrix} \tag{6}$$

Ply Integral Equation

Each individual ply, with section Ω_e bounded by the contour Γ_e, is subjected on the lateral surface to the interlaminar tractions $\boldsymbol{t}$ satisfying the equilibrium equations (5) and depending on x_1 and x_2 only. Let us consider a particular solution of the elasticity problem relative to the ply loaded by fictitious body forces $\boldsymbol{f}_j = \boldsymbol{f}_j(x_1, x_2)$, which do not vary along the longitudinal axis z. Let this particular solution, referred in the following as the problem fundamental solution, be associated with a displacement field $\boldsymbol{u}_j(x_1, x_2)$ that satisfies the equilibrium equations[33,34,36]

$$\boldsymbol{D}^T \boldsymbol{E}\boldsymbol{D}\boldsymbol{u}_j + \boldsymbol{f}_j = \boldsymbol{0} \tag{7}$$

in Ω_e. According to the form of the compatibility and constitutive equations employed in the preceding section, let $\boldsymbol{\varepsilon}_j$, ε_{33j} and $\boldsymbol{\sigma}_j$, σ_{33j} be the strain and stress field related to $\boldsymbol{u}_j$ and again let $\boldsymbol{t}_j$ be the tractions. The boundary integral equations are obtained by applying the Betti reciprocal work theorem and using the fundamental solution given by Eq. (7). Taking Eqs. (4–7) into account, we have the following integral relation[34]:

$$\int_{\Gamma_e} (\boldsymbol{t}_j^T \boldsymbol{s} - \boldsymbol{u}_j^T \boldsymbol{t})\, \mathrm{d}\Gamma_e + \int_{\Omega_e} \boldsymbol{f}_j^T \boldsymbol{s}\, \mathrm{d}\Omega_e = \int_{\Omega_e} (\boldsymbol{\sigma}_j^T \boldsymbol{D}\boldsymbol{s} - \boldsymbol{\varepsilon}_j^T \boldsymbol{\sigma})\, \mathrm{d}\Omega_e \tag{8}$$

Equations (1) imply

$$\boldsymbol{D}\boldsymbol{s} = \boldsymbol{D}\boldsymbol{u} \tag{9}$$

and so Eq. (8) is satisfied if one simultaneously has

$$\int_{\Gamma_e} (\boldsymbol{t}_j^T \boldsymbol{u} - \boldsymbol{u}_j^T \boldsymbol{t})\, \mathrm{d}\Gamma_e + \int_{\Omega_e} \boldsymbol{f}_j^T \boldsymbol{u}\, \mathrm{d}\Omega_e = \int_{\Omega_e} (\boldsymbol{\sigma}_j^T \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_j^T \boldsymbol{\sigma})\, \mathrm{d}\Omega_e \tag{10}$$

$$\int_{\Gamma_e} \boldsymbol{t}_j^T \boldsymbol{s}_0\, \mathrm{d}\Gamma_e + \int_{\Omega_e} \boldsymbol{f}_j^T \boldsymbol{s}_0\, \mathrm{d}\Omega_e = \boldsymbol{0} \tag{11}$$

where

$$\boldsymbol{s}_0^T = [-\vartheta x_2 \quad \vartheta x_1 \quad 0] \tag{12}$$

The displacement system $\boldsymbol{s}_0$ identifies a rigid motion rotation in the $x_1 x_2$ plane, and the body forces $\boldsymbol{f}_j$ are equilibrated by the tractions $\boldsymbol{t}_j$:

$$\int_{\Gamma_e} \boldsymbol{t}_j^T\, \mathrm{d}\Gamma_e + \int_{\Omega_e} \boldsymbol{f}_j^T\, \mathrm{d}\Omega_e = \boldsymbol{0} \tag{13}$$

Thus, the reciprocal work statement (11) is identically satisfied and the integral representation for the ply elastic behavior is given by

Eq. (10) only. Taking into account the constitutive equations (3a), Eq. (10) becomes

$$\int_{\Gamma_e} (\boldsymbol{t}_j^T \boldsymbol{u} - \boldsymbol{u}_j^T \boldsymbol{t})\, \mathrm{d}\Gamma_e + \int_{\Omega_e} \boldsymbol{f}_j^T \boldsymbol{u}\, \mathrm{d}\Omega_e = -\vartheta \int_{\Omega_e} \boldsymbol{\varepsilon}_j^T \boldsymbol{Q}_1 \boldsymbol{X}\, \mathrm{d}\Omega_e \qquad (14)$$

To obtain the integral equations governing the problem, the singular fundamental solution due to a concentrated load distributed along a line parallel to the longitudinal axis and applied in the j direction is used. The load $\boldsymbol{f}_j$ is, thus, characterized as

$$\boldsymbol{f}_j = \boldsymbol{c}_j \delta(P - P_0) \qquad (15)$$

in which P_0 is the load application point in the ply cross section and $\delta(P - P_0)$ is the Dirac function. With this assumption, from Eq. (14) one obtains

$$\boldsymbol{c}_j^T \boldsymbol{u}(P_0) + \int_{\Gamma_e} (\boldsymbol{t}_j^T \boldsymbol{u} - \boldsymbol{u}_j^T \boldsymbol{t})\, \mathrm{d}\Gamma_e = -\vartheta \int_{\Omega_e} \boldsymbol{\varepsilon}_j^T \boldsymbol{Q}_1 \boldsymbol{X}\, \mathrm{d}\Omega_e \qquad (16)$$

where

$$\boldsymbol{c}_j = \int_{\Omega_e} \boldsymbol{f}_j\, \mathrm{d}\Omega_e = -\int_{\Gamma_e} \boldsymbol{t}_j\, \mathrm{d}\Gamma_e \qquad (17)$$

Equation (16) is the integral equation governing the behavior of the individual ply and, together with the interface continuity conditions and the prescribed boundary data, it should be sufficient to characterize the laminate elastic response. The domain integrals involved in evaluating the right-hand side of Eq. (16) can be converted into boundary integrals over Γ_e. To obtain this goal, let us consider a particular solution $\bar{\boldsymbol{s}}$ of the ply governing equations

$$\bar{s}_1 = \bar{u}_1 - \vartheta x_2 z \qquad (18a)$$

$$\bar{s}_2 = \bar{u}_2 + \vartheta x_1 z \qquad (18b)$$

$$\bar{s}_3 = \bar{u}_3 \qquad (18c)$$

By writing Eq. (16) for this particular solution, one obtains

$$\boldsymbol{c}_j^T \bar{\boldsymbol{u}}(P_0) + \int_{\Gamma_e} (\boldsymbol{t}_j^T \bar{\boldsymbol{u}} - \boldsymbol{u}_j^T \bar{\boldsymbol{t}})\, \mathrm{d}\Gamma_e = -\vartheta \int_{\Omega_e} \boldsymbol{\varepsilon}_j^T \boldsymbol{Q}_1 \boldsymbol{X}\, \mathrm{d}\Omega_e \qquad (19)$$

which allows one to reduce Eq. (16) as follows:

$$\boldsymbol{c}_j^T \boldsymbol{u}(P_0) + \int_{\Gamma_e} (\boldsymbol{t}_j^T \boldsymbol{u} - \boldsymbol{u}_j^T \boldsymbol{t})\, \mathrm{d}\Gamma_e = \boldsymbol{c}_j^T \bar{\boldsymbol{u}}(P_0) + \int_{\Gamma_e} (\boldsymbol{t}_j^T \bar{\boldsymbol{u}} - \boldsymbol{u}_j^T \bar{\boldsymbol{t}})\, \mathrm{d}\Gamma_e \qquad (20)$$

Equation (20) represents the final form of the integral equation employed for the problem solution, and it evidences the boundary-only character of the model proposed.

BEM Model

The boundary integral formulation discussed in the preceding section is solved by the classical BEM. Once the ply section boundary has been discretized by boundary elements, the unknown displacement functions $\boldsymbol{u}$ and tractions $\boldsymbol{t}$ can be expressed using their nodal values

$$\boldsymbol{u} = \boldsymbol{N}\boldsymbol{\delta} \qquad (21a)$$

$$\boldsymbol{t} = \boldsymbol{N}\boldsymbol{p} \qquad (21b)$$

where $\boldsymbol{N}$ is a matrix of properly selected shape functions and $\boldsymbol{\delta}$ and $\boldsymbol{p}$ denote vectors of displacement function and traction nodal values, respectively. Considering the three independent load directions $j = 1, 2, 3$, the discretized boundary integral equations are obtained for the source point P_0:

$$\boldsymbol{c}^*\boldsymbol{u}(P_0) + \int_{\Gamma_e} \boldsymbol{t}^*\boldsymbol{N}\, \mathrm{d}\Gamma_e \boldsymbol{\delta} - \int_{\Gamma_e} \boldsymbol{u}^*\boldsymbol{N}\, \mathrm{d}\Gamma_e \boldsymbol{p} = \boldsymbol{c}^*\bar{\boldsymbol{u}}(P_0) + \int_{\Gamma_e} (\boldsymbol{t}^*\bar{\boldsymbol{u}} - \boldsymbol{u}^*\bar{\boldsymbol{t}})\, \mathrm{d}\Gamma_e \qquad (22)$$

where

$$\boldsymbol{c}^* = [c_{ij}]^T, \qquad \boldsymbol{u}^* = [u_{ij}]^T, \qquad \boldsymbol{t}^* = [t_{ij}]^T, \qquad i, j = 1, 2, 3 \qquad (23)$$

and u_{ij} and t_{ij} are the components of displacements and tractions of the three fundamental solutions (see Appendix). The particular solution $\bar{\boldsymbol{s}}$ needed to express the right-hand side of Eq. (22) can be sought in the form of polynomials, and hence, according to the anisotropic elasticity theory, one has[21]

$$\bar{u}_1 = \tfrac{1}{2}K_{11}x_1^2 + K_{12}x_1x_2 + K_{16}x_1 + \tfrac{1}{2}(K_{32} - K_{21})x_2^2 + \tfrac{1}{2}K_{36}x_2 \qquad (24a)$$

$$\bar{u}_2 = K_{21}x_1x_2 + \tfrac{1}{2}K_{22}x_2^2 + K_{26}x_2 + \tfrac{1}{2}(K_{31} - K_{12})x_1^2 + \tfrac{1}{2}K_{36}x_1 \qquad (24b)$$

$$\bar{u}_3 = \tfrac{1}{2}K_{41}x_1^2 + \tfrac{1}{2}(K_{42} + \vartheta)x_1x_2 + K_{46}x_1 + \tfrac{1}{2}K_{52}x_2^2 + K_{56}x_2 \qquad (24c)$$

In the preceding expressions, one has to set

$$K_{j1} = 2C_{j1}a_1 + 6C_{j2}a_2 - 2C_{j3}a_3 + C_{j4}a_4 - 2C_{j5}a_5 \qquad (25a)$$

$$K_{j2} = 6C_{j1}a_6 + 2C_{j2}a_3 - 2C_{j3}a_1 + 2C_{j4}a_7 - C_{j5}a_4 \qquad (25b)$$

$$K_{j6} = 2C_{j1}a_8 + 2C_{j2}a_9 - C_{j3}a_{10} + C_{j4}a_{11} - C_{j5}a_{12} \qquad (25c)$$

where the modified compliances C_{rs} are defined as

$$[C_{rs}] = \boldsymbol{E}^{-1} \qquad (26)$$

The constants $a_i (i = 1, \ldots, 12)$ appearing in Eqs. (25) can be arbitrarily assigned provided that they satisfy the following equation:

$$-2(C_{15} + C_{43})a_1 - 6C_{25}a_2 + 2(C_{24} + C_{53})a_3 - 2C_{54}a_4 + 2C_{55}a_5 + 6C_{14}a_6 + 2C_{44}a_7 = -2\vartheta \qquad (27)$$

Collocating Eq. (22) at each nodal point, a set of linear algebraic equations is obtained that describes the boundary behavior of the considered ply. By the enforcement of the interfacial continuity conditions and the prescribed boundary data and considering all of the layers that constitute the laminate, one deduces all of the relations desired to solve the problem. The fundamental requirement of vanishing shear stress at the free edge interface junction can be taken into account by means of linear combinations of the equations written at the interface corners.[34] Once the boundary solution is determined, Eq. (20) gives the displacements at the laminate generic point P_0. The strains at the generic ply point P_0 can be calculated in a pointwise fashion by differentiation of Eq. (20) with respect to the source point P_0, and consequently, the stresses can be obtained from the following integral relations:

$$\boldsymbol{\sigma} = \boldsymbol{E}\left\{\int_{\Gamma_e} \boldsymbol{T}^*(\boldsymbol{u} - \bar{\boldsymbol{u}})\, \mathrm{d}\Gamma_e - \int_{\Gamma_e} \boldsymbol{U}^*(\boldsymbol{t} - \bar{\boldsymbol{t}})\, \mathrm{d}\Gamma_e + \bar{\boldsymbol{\varepsilon}}(P_0)\right\} + \boldsymbol{Q}_1\boldsymbol{X}\vartheta \qquad (28a)$$

$$\sigma_{33} = \boldsymbol{Q}_2\left\{\int_{\Gamma_e} \boldsymbol{T}^*(\boldsymbol{u} - \bar{\boldsymbol{u}})\, \mathrm{d}\Gamma_e - \int_{\Gamma_e} \boldsymbol{U}^*(\boldsymbol{t} - \bar{\boldsymbol{t}})\, \mathrm{d}\Gamma_e + \bar{\boldsymbol{\varepsilon}}(P_0)\right\} - E_{46}x_2\vartheta \qquad (28b)$$

The kernels $\boldsymbol{T}^*$ and $\boldsymbol{U}^*$ (see Appendix) are obtained by applying the strain operator $\boldsymbol{D}^* = \boldsymbol{D}\boldsymbol{c}^{*-1}$ to $\boldsymbol{t}^*$ and $\boldsymbol{u}^*$, respectively, whereas $\bar{\boldsymbol{\varepsilon}}$ is obtained by applying the strain operator $\boldsymbol{D}$ to the particular displacement function vector $\bar{\boldsymbol{u}}$.

Numerical Applications

The method was applied to some typical laminate configurations to determine the elastic response under twisting loads. Because of the crucial role played by the interlaminar stresses in laminate failure, particular care was devoted to the calculation of the interlaminar stress distributions for which some results are available in the literature.[23] The laminates considered for the analysis consist of plies with rectangular cross section having thickness h and width $2b = 16h$. The material properties in gigapascal are those of graphite/epoxy fiber-reinforced laminae:

$$E_{LL} = 137.9, \qquad E_{TT} = E_{SS} = 14.5 \tag{29}$$
$$G_{LT} = G_{LS} = G_{TS} = 5.9, \qquad \nu_{LT} = \nu_{LS} = \nu_{TS} = 0.21$$

where the subscripts L, T, and S refer to the along fiber, thickness, and width directions. The numerical results presented in this section were obtained by using an expressly developed computer code. The ply's influence coefficients are calculated through an adaptive Gaussian quadrature scheme, which allows one to resolve the kernel singularities numerically.[37] The contribution due to the twisting curvature θ, i.e., the right-hand side of Eq. (22), is calculated by using a particular solution $\bar{s}$, which was obtained from Eqs. (24) and (25) specifying

$$a_5 = -(\vartheta / C_{55}) \tag{30}$$

and all of the other constants a_i equal to zero. The formulation proposed exactly satisfies all of the elasticity relations and boundary conditions, and hence, the numerical solution given by the BEM is actually consistent. Therefore, the accuracy of the resulting solution depends on the mesh refinement only. The mesh arrangement used to obtain the results presented is shown in Fig. 2. The boundary of each individual ply is discretized by 104 boundary elements with linear interpolation functions. By virtue of the symmetries inherent in the laminates under consideration and in the loading conditions, only a quarter of the laminate cross section may be considered in the analysis with the appropriate boundary symmetry conditions imposed along the x_1 and x_2 axes. Here, also for standard symmetric laminates, the entire cross section was discretized with the aim of testing confidence in the method in view of more complex analyses. Prescribed displacement conditions on the displacement functions $\boldsymbol{u}$ must be superimposed to eliminate the unconstrained rigid motion of the cross section.

In Fig. 3 the interlaminar stresses at the top interface of the $[45/-45]_S$ configuration are shown for a half-width of the laminate by virtue of the stress field symmetries. The comparison with the results of Yin[23] proves the accuracy of the present solution and suggests

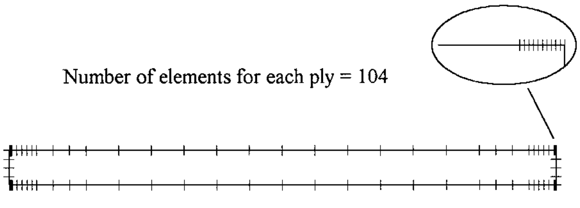

Fig. 2 Boundary element mesh for each individual ply.

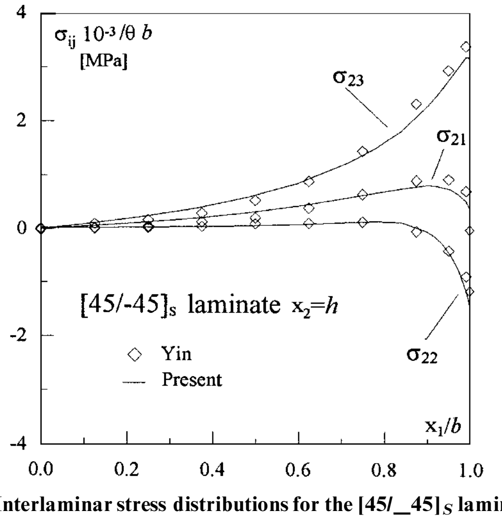

Fig. 3 Interlaminar stress distributions for the $[45/-45]_S$ laminate in torsion.

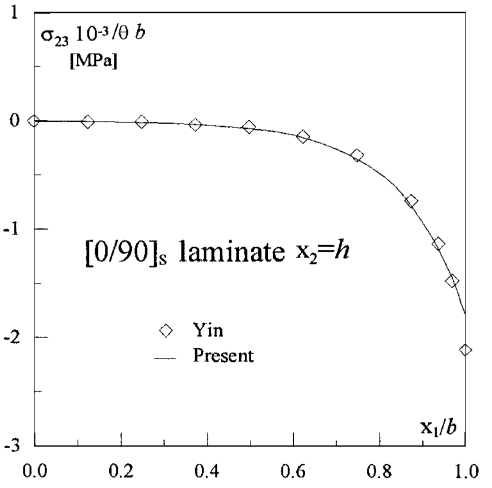

Fig. 4 Interlaminar stress distribution for the $[0/90]_S$ laminate in torsion.

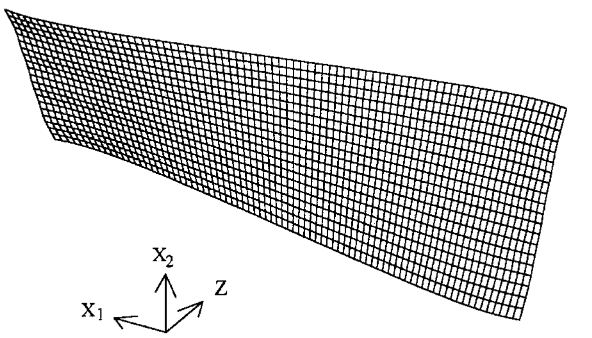

Fig. 5 Cross section deformed shape for the $[0/90]_S$ laminate in torsion, $z = 0$.

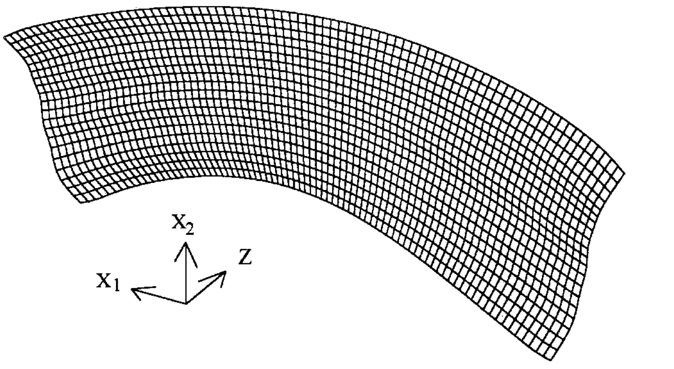

Fig. 6 Cross section deformed shape for the $[45/-45]_S$ laminate in torsion, $z = 0$.

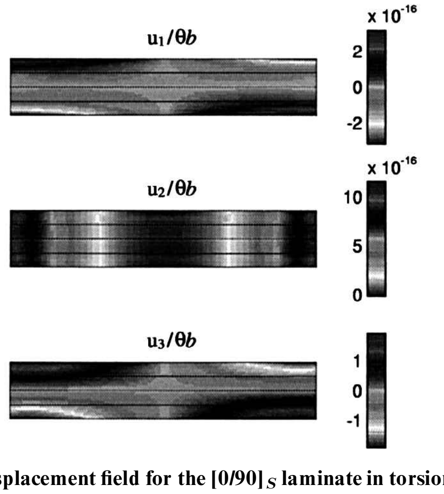

Fig. 7 Displacement field for the $[0/90]_S$ laminate in torsion, $z = 0$.

that the integral equation solution better describes the stress field behavior near the free edge.

In Fig. 4 the sole interlaminar stress σ_{23} existing in the $[0/90]_S$ configuration is shown as a function of the width at the 0/90 interface. Again the comparison with the results of Yin[23] confirms the accuracy of the method. The efficiency exhibited by the calculations worked out and the accuracy of the results obtained meaningful evidence of computational advantages of the present method. In the case of four-layer laminates, execution of the computer code takes about 50 s on a personal computer. Moreover, the comparison of the solution presented with those already available establishes confidence in the method and the computer code for their use in the analysis of general configurations. For the laminates considered, a complete analysis of the elastic response was also carried out calculating the displacement and stress field in a pointwise fashion by means of Eqs. (22) and the discrete form of Eqs. (28).

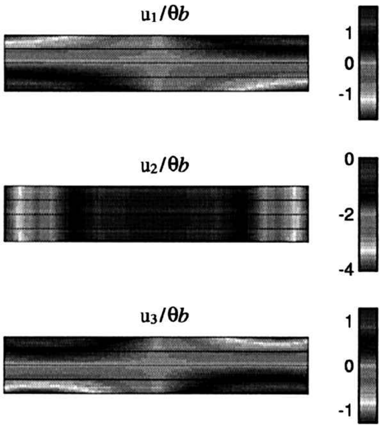

Fig. 8 Displacement field for the [45/_45]$_S$ laminate in torsion, $z = 0$.

In Figs. 5 and 6 the cross section deformed shapes at $z = 0$ for the two, four-ply, symmetric $[0/90]_S$ and [45/_45]$_S$ laminates are shown, respectively. These results, like those presented in the following for the stress fields, were obtained considering, for each ply, 96 internal points where the displacement and stress integral relations were numerically evaluated through Gaussian quadrature starting from the determined boundary solution.

The displacement component distributions for the same laminates are shown in Figs. 7 and 8. The stress fields are depicted in Figs. 9 and 10. The interlaminar stress distribution near the free edge indicates that a twisting deformation θ produces a very large mode III interlaminar stress. This trend is susceptible to delamination failure under the shear fracture modes. The proposed method can be straightforwardly extended to the analysis of damage initiation and progression releasing the interface continuity conditions. The efficiency demonstrated in the calculations of the elastic response for the whole laminate suggests that the alternative method presented here for the analysis of composite laminates in torsion could be successfully applied to more complex and realistic cases, coupling high accuracy and considerable computational convenience.

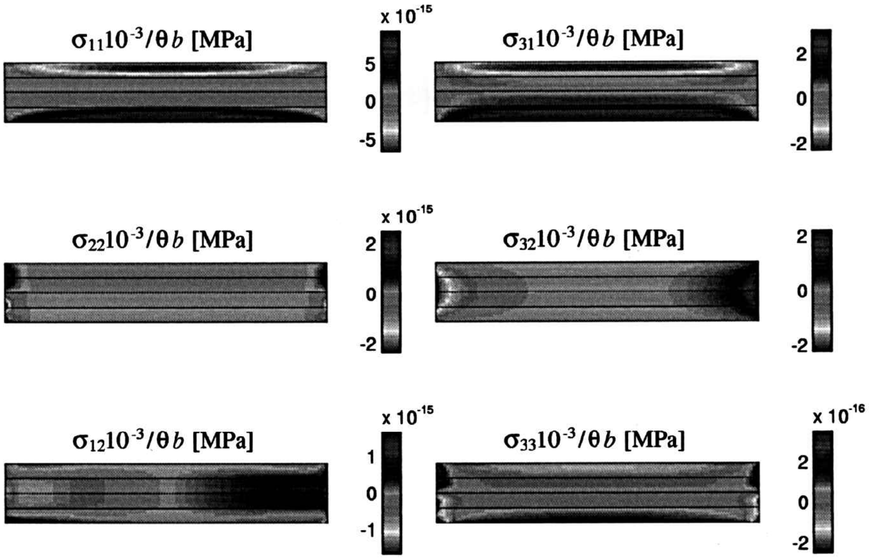

Fig. 9 Stress field for the $[0/90]_S$ laminate in torsion, $z = 0$.

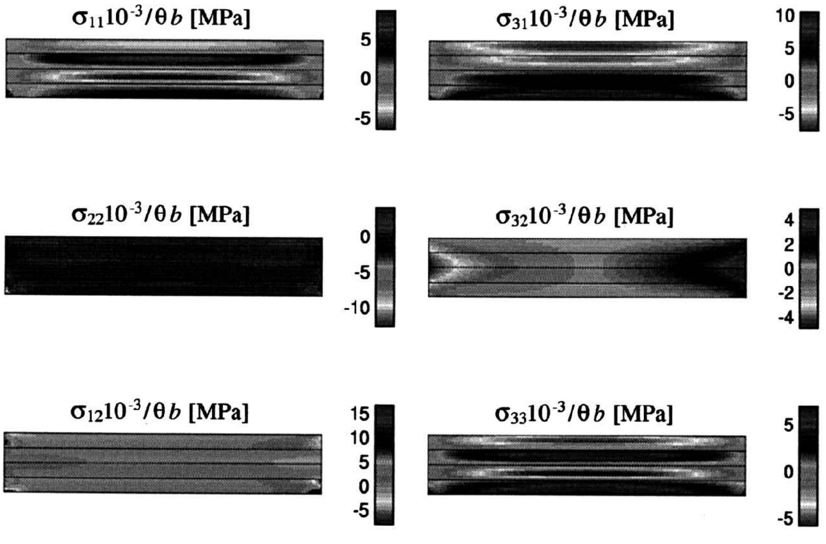

Fig. 10 Stress field for the [45/_45]$_S$ laminate in torsion, $z = 0$.

Conclusions

The solution of the elasticity problem for a general composite laminate under twisting loads applied at the laminate's ends is presented. The formulation of the problem is based on the integral equation theory, and the governing integral equations are deduced. The model obtained using this alternative approach solves exactly the problem by virtue of the complete fulfillment of all of the elasticity relations and boundary conditions involved in the problem's description. As a consequence, the numerical solution of the formulation through the BEM leads to solutions for this class of composite laminates that are consistent because they depend on the mesh refinement only. Some solutions for typical laminate configurations were obtained, and the calculations performed evidenced the presence of meaningful computational advantages of the proposed method. Moreover, the comparison of the present results with those already available in the literature proves the accuracy of the solutions obtained and establishes confidence in the method for its application to general configurations. In conclusion, the exhibited features indicate the proposed alternative approach as a powerful tool to investigate the elastic response of laminated composites in torsion.

Appendix: Fundamental Solutions

The kernels involved in the integral equations for displacements [Eq. (20)] and stresses [Eqs. (28)] are obtained from the solution of the following equilibrium equations:

$$\boldsymbol{D}^T \boldsymbol{E} \boldsymbol{D} \boldsymbol{u}_j + \boldsymbol{c}_j \delta(P - P_0) = \boldsymbol{0} \tag{A1}$$

in Ω_e, where the vector $\boldsymbol{c}_j$ contains the load components and $\delta(P - P_0)$ indicates the Dirac function. According to the anisotropic elasticity theory,[35] the solutions of Eq. (A1), i.e., the problem fundamental solutions, depend on the roots of the characteristic equation

$$a\lambda^3 + b\lambda^2 + c\lambda + d = 0 \tag{A2}$$

where

$$a = C_{11}C_{44} - C_{14}^2 \tag{A3}$$

$$b = 2C_{14}(C_{24} + C_{35}) - C_{11}C_{55} - C_{44}(2C_{12} + C_{33}) \tag{A4}$$

$$c = C_{22}C_{44} + C_{55}(2C_{12} + C_{33}) - (C_{24} + C_{35})^2 \tag{A5}$$

$$d = -C_{22}C_{55} \tag{A6}$$

In the case of distinct and positive roots λ_i, one has

$$\boldsymbol{u}_j = \begin{bmatrix} u_{1j} \\ u_{2j} \\ u_{3j} \end{bmatrix} = \begin{bmatrix} \varphi_1 B_{11} & \varphi_2 B_{12} & \varphi_3 B_{13} \\ \psi_1 B_{21} & \psi_2 B_{22} & \psi_3 B_{23} \\ \varphi_1 B_{31} & \varphi_2 B_{32} & \varphi_3 B_{33} \end{bmatrix} \begin{bmatrix} A_{1j} \\ A_{2j} \\ A_{3j} \end{bmatrix} \tag{A7}$$

where for $j = 1$ and 3, the functions φ_i and ψ_i are given by

$$\varphi_i(P, P_0) = -(\ln r_i)/2\pi \tag{A8}$$

$$\psi_i(P, P_0) = -\left[\tan^{-1}\left(\sqrt{\lambda_i}\, y/x\right)\right]\Big/2\pi\sqrt{\lambda_i} \tag{A9}$$

whereas for $j = 2$, one has to set

$$\varphi_i(P, P_0) = -\left[\tan^{-1}\left(\sqrt{\lambda_i}\, y/x\right)\right]\sqrt{\lambda_i}\Big/2\pi \tag{A10}$$

$$\psi_i(P, P_0) = (\ln r_i)/2\pi \tag{A11}$$

In the preceding relations, the following definitions are valid:

$$x = x_1(P) - x_1(P_0) \tag{A12}$$

$$y = x_2(P) - x_2(P_0) \tag{A13}$$

$$r_i = \sqrt{x^2 + \lambda_i y^2} \tag{A14}$$

where P is the observed point and P_0 is the source point where the load is applied. Again one has to set

$$B_{1i} = C_{11} - C_{12}/\lambda_i + C_{14}\gamma_i \tag{A15}$$

$$B_{2i} = C_{12} - C_{22}/\lambda_i + C_{24}\gamma_i \tag{A16}$$

$$B_{3i} = C_{14} - C_{24}/\lambda_i + C_{44}\gamma_i \tag{A17}$$

where

$$\gamma_i = \frac{C_{14}\lambda_i - C_{24} - C_{35}}{C_{55} - C_{44}\lambda_i} \tag{A18}$$

For $j = 1$ and 3, the coefficients A_{ij} are given by

$$\boldsymbol{A}_j = \begin{bmatrix} A_{1j} \\ A_{2j} \\ A_{3j} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{\lambda_1}} & \frac{1}{\sqrt{\lambda_2}} & \frac{1}{\sqrt{\lambda_3}} \\ \frac{B_{21}}{\sqrt{\lambda_1}} & \frac{B_{22}}{\sqrt{\lambda_2}} & \frac{B_{23}}{\sqrt{\lambda_3}} \\ \frac{\gamma_1}{\sqrt{\lambda_1}} & \frac{\gamma_2}{\sqrt{\lambda_2}} & \frac{\gamma_3}{\sqrt{\lambda_3}} \end{bmatrix}^{-1} \begin{bmatrix} c_{1j} \\ 0 \\ c_{3j} \end{bmatrix} \tag{A19}$$

whereas for $j = 2$, they are given by

$$\boldsymbol{A}_2 = \begin{bmatrix} A_{12} \\ A_{22} \\ A_{32} \end{bmatrix} = \begin{bmatrix} B_{11}\sqrt{\lambda_1} & B_{12}\sqrt{\lambda_2} & B_{13}\sqrt{\lambda_3} \\ \frac{1}{\sqrt{\lambda_1}} & \frac{1}{\sqrt{\lambda_2}} & \frac{1}{\sqrt{\lambda_3}} \\ B_{31}\sqrt{\lambda_1} & B_{32}\sqrt{\lambda_2} & B_{33}\sqrt{\lambda_3} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ c_{22} \\ 0 \end{bmatrix} \tag{A20}$$

The traction kernels can be expressed as

$$\boldsymbol{t}_j = \begin{bmatrix} t_{1j} \\ t_{2j} \\ t_{3j} \end{bmatrix} = \begin{bmatrix} \Lambda_1 & \Lambda_2 & \Lambda_3 \\ \Pi_1 & \Pi_2 & \Pi_3 \\ \gamma_1\Lambda_1 & \gamma_2\Lambda_2 & \gamma_3\Lambda_3 \end{bmatrix} \begin{bmatrix} A_{1j} \\ A_{2j} \\ A_{3j} \end{bmatrix} \tag{A21}$$

where for $j = 1$ and 3,

$$\Lambda_i = -(x\alpha_1 + y\alpha_2)\Big/2\pi r_i^2 \tag{A22}$$

$$\Pi_i = (x\alpha_2 - \lambda_i y\alpha_1)\Big/2\pi\lambda_i r_i^2 \tag{A23}$$

whereas for $j = 2$,

$$\Lambda_i = (\lambda_i y\alpha_1 - x\alpha_2)\Big/2\pi r_i^2 \tag{A24}$$

$$\Pi_i = -(x\alpha_1 + y\alpha_2)\Big/2\pi r_i^2 \tag{A25}$$

For P_0 belonging to Ω_e the fundamental solutions are, in general, obtained by setting $c_{ij}(P_0) = \delta_{ij}$, where δ_{ij} is the Kronecker symbol. Upon this condition, if P_0 lies on a smooth boundary, one has $c_{ij}(P_0) = \delta_{ij}/2$. The kernels $\boldsymbol{U}* = [U_{ij}]$ and $\boldsymbol{T}* = [T_{ij}]$ involved in the integral relations (28), providing the stresses at the source field point, can be defined in terms of their columns as follows:

$$\boldsymbol{U}_j^* = \begin{bmatrix} U_{1j}^* \\ U_{2j}^* \\ U_{3j}^* \\ U_{4j}^* \\ U_{5j}^* \end{bmatrix} = \begin{bmatrix} B_{11}\zeta_1 & B_{12}\zeta_2 & B_{13}\zeta_3 \\ B_{21}\zeta_1 & B_{22}\zeta_2 & B_{23}\zeta_3 \\ (B_{11}\lambda_1 - B_{21})\chi_1 & (B_{12}\lambda_2 - B_{22})\chi_2 & (B_{13}\lambda_3 - B_{23})\chi_3 \\ B_{31}\zeta_1 & B_{32}\zeta_2 & B_{33}\zeta_3 \\ B_{31}\lambda_1\chi_1 & B_{32}\lambda_2\chi_2 & B_{33}\lambda_3\chi_3 \end{bmatrix} \begin{bmatrix} A_{1j} \\ A_{2j} \\ A_{3j} \end{bmatrix} \tag{A26}$$

$$
\boldsymbol{T}_j^* = \begin{bmatrix} T_{1j}^* \\ T_{2j}^* \\ T_{3j}^* \\ T_{4j}^* \\ T_{5j}^* \end{bmatrix} = \begin{bmatrix} \eta_1 & \eta_2 & \eta_3 \\ -\eta_1 & -\eta_2 & -\eta_3 \\ \dfrac{\lambda_1 + 1}{\lambda_1}\mu_1 & \dfrac{\lambda_2 + 1}{\lambda_2}\mu_2 & \dfrac{\lambda_3 + 1}{\lambda_3}\mu_3 \\ \gamma_1\eta_1 & \gamma_2\eta_2 & \gamma_3\eta_3 \\ \gamma_1\mu_1 & \gamma_2\mu_2 & \gamma_3\mu_3 \end{bmatrix} \begin{bmatrix} A_{1j} \\ A_{2j} \\ A_{3j} \end{bmatrix} \tag{A27}
$$

where for $j = 1$ and 3, one has

$$
\eta_i = -\frac{(x^2 - \lambda_i y^2)\alpha_1 + 2xy\alpha_2}{r_i^4} \tag{A28}
$$

$$
\mu_i = -\frac{2\lambda_i xy\alpha_1 - (x^2 - \lambda_i y^2)\alpha_2}{r_i^4} \tag{A29}
$$

$$
\zeta = x/r_i^2 \tag{A30}
$$

$$
\chi_i = y/r_i^2 \tag{A31}
$$

whereas for $j = 2$, one has

$$
\eta_i = \frac{2\lambda_i xy\alpha_1 - (x^2 - \lambda_i y^2)\alpha_2}{r_i^4} \tag{A32}
$$

$$
\mu_i = -\frac{\lambda_i(x^2 - \lambda_i y^2)\alpha_1 + 2xy\alpha_2}{r_i^4} \tag{A33}
$$

$$
\zeta = -\lambda_i(y/r_i^2) \tag{A34}
$$

$$
\chi_i = x/r_i^2 \tag{A35}
$$

In the case of a cross-ply composite laminate, by virtue of the material symmetry in each ply, the fundamental solutions are recovered through a suitable limit procedure applied at the preceding relations.[36]

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